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A Lyapunov approach to the stability of fractional differential equations

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ABSTRACT

Lyapunov stability of fractional differential equations is addressed in this paper. The key concept is the frequency distributed fractional integrator model, which is the basis for a global state space model of FDEs. Two approaches are presented: the direct one is intuitive but it leads to a large dimension parametric problem while the indirect one, which is based on the continuous frequency distribution, leads to a parsimonious solution. Two examples, with linear and nonlinear FDEs, exhibit the main features of this new methodology.

1. Introduction

Stability of linear Ordinary Differential Equations (ODEs) can be analyzed by Lyapunov's technique [14], though Routh–Hurwitz criterion [19,10,8] is more adapted to the formulation of parametric stability conditions. The main interest of Lyapunov's approach is to define Linear Matrix Inequalities (LMIs) conditions [3], which can be used for example in robust controller synthesis. But it is also well known that Lyapunov's technique is the fundamental tool to analyze the stability of nonlinear systems [12].

In spite of their increasing interest in the modeling of diffusion processes [1,18] and in the synthesis of robust control laws [17], Fractional Differential Equations (FDEs) have not yet received the same attention as ODEs in the investigation of their stability.

A recent paper by Sabatier et al. [20] provides a good survey of the methods available to analyze the stability of FDEs. In the linear case, with the hypothesis of commensurate order systems, the main results are Matignon's theorem [15] and input/output stability [2]. Recently, a frequency approach to the stability of FDEs, without explicit computation of system poles, has been proposed [1,22].

Lyapunov's approach, in spite of its interest in the formulation of LMI conditions, has not yet received satisfactory solutions, and more specifically in the nonlinear case [20]. Nevertheless, Mittag–Leffler's stability of nonlinear FDEs has been addressed in a recent paper [13]. The main drawback of these contributions is to rely on a nonsatisfactory definition of FDE state variables and thus of their state space representation. An indirect approach, without explicit formulation of a Lyapunov function, has been used by Sabatier et al. [20] to formulate LMI conditions related to commensurate order FDEs.

In this paper, we propose the application of Lyapunov's method to linear and nonlinear FDEs thanks to the definition of a specific Lyapunov function. Our approach relies fundamentally on the concept of a fractional integration operator characterized by a continuous frequency distributed modal law [23]. The corresponding state space representation allows the definition of an elementary monochromatic Lyapunov function $v(\omega,t)$ which leads to

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the FDE Lyapunov function $V(t)$, by integration of all the monochromatic contributions on the whole spectral range. The main interest of this Lyapunov function is that it leads to a parsimonious parametric stability condition, which would not be possible with a direct application of Lyapunov usual methodology.

The paper is organized as follows. Definitions related to fractional integration and FDEs are reminded in Section 2. The frequency distributed model of the fractional integrator and the corresponding global state space model of the FDE are analyzed in Sections 3 and 4. A direct approach to Lyapunov stability is presented in Section 5. Finally, an indirect approach, based on the continuous frequency distributed fractional integrator model is proposed in Section 6.

2. Definitions related to fractional systems

2.1. Fractional derivation and integration

Fractional integration is defined by the Riemann–Liouville integral [17,18].

The nth order integral (n real positive) of the function $f(t)$ is defined by the relation

$$I^n(f(t)) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau \tag{1}$$

where

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \tag{2}$$

is the gamma function.

$I^n(f(t))$ is interpreted as the convolution of the function $f(t)$with the impulse response

$$h_n(t) = \frac{t^{n-1}}{\Gamma(n)} \tag{3}$$

of the fractional integration operator whose Laplace transform is

$$I^n(s) = L\{h_n(t)\} = \frac{1}{s^n} \tag{4}$$

Fractional derivation is the dual operation of the fractional integration.

Consider the fractional integration operator $I^n(s)$ whose input and output are, respectively, $x(t)$ and $y(t)$.

Then

$$y(t) = I^n(x(t)) \tag{5}$$

or

$$Y(s) = \frac{1}{s^n} X(s) \tag{6}$$

Reciprocally, $x(t)$ is the nth order fractional derivative of $y(t)$ (n real positive) defined as

$$x(t) = D^n(y(t)) \tag{7}$$

or

$$X(s) = s^n Y(s) \tag{8}$$

where s^n represents the Laplace transform of the fractional derivation operator, with the assumption of zero initial conditions.

2.2. Fractional differential equation (FDE)

Consider the general linear FDE ($M \le N$)

$$D^{m_N}(y(t)) + a_{N-1}D^{m_{N-1}}(y(t)) + \cdots + a_1 D^{m_1}(y(t)) + a_0 y(t)$$
$$= b_M D^{m_M}(u(t)) + \cdots + b_1 D^{m_1}(u(t)) + b_0 u(t) \tag{9}$$

whose transfer function is (with the assumption of zero initial conditions)

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^{m_1} + \cdots + b_M s^{m_M}}{a_0 + a_1 s^{m_1} + \cdots + a_{N-1}s^{m_{N-1}} + s^{m_N}} = \frac{B(s)}{A(s)} \tag{10}$$

The fractional derivation orders:

$$m_1 < m_2 < \cdots < m_N \tag{11}$$

are real positive numbers; they are called external or explicit orders. It is necessary to define internal or implicit derivation orders such as

$$\begin{aligned} n_1 &= m_1 \\ &\vdots \\ n_i &= m_i - m_{i-1} \\ &\vdots \\ n_N &= m_N - m_{N-1} \end{aligned} \tag{12}$$

Remark the implicit orders n_i have been derived from a composition of fractional derivatives (or equivalently of fractional integrators), which has been analyzed by Diethelm and Ford in Lemma 2.3 [5].

3. Fractional integration operator

The fractional integration operator $I^n(s)$ is the key element for FDE simulation. However, the realization of $I^n(s)$, either in analog or numerical form, is not a simple problem, as in the integer order case. It is possible to consider the frequency [21] and the time approaches. Let us remind the time approach synthesis [23].

3.1. Principle

Diffusive representation, used by Heleschewitz and Matignon [9] and Montseny [16] provide the theoretical basis for a time approximation of $I^n(s)$.

Consider a linear system such as

$$x(t) = h(t)v(t) \tag{13}$$

where $h(t)$ is its impulse response and $\mu(\omega)$ is called the diffusive representation (or frequency weighting function) of the impulse response $h(t)$. $h(t)$ and $\mu(\omega)$ verify the pseudo Laplace transform definition [16]

$$h(t) = \int_0^\infty \mu(\omega) e^{-\omega t}\,d\omega \tag{14}$$

A continuous frequency weighted state space model is associated to $\mu(\omega)$, according to

$$\begin{cases} \dfrac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + v(t) \\ x(t) = \int_0^\infty \mu(\omega) z(\omega,t)\,d\omega \end{cases} \tag{15}$$

For the fractional integration operator

$$I^n(s) = \frac{1}{s^n} \text{ with } 0<n<1,$$

$$h(t) = \frac{t^{n-1}}{\Gamma(n)}$$

and

$$\mu(\omega) = \frac{\sin(n\pi)}{\pi}\omega^{-n} \tag{16}$$

3.2. Discrete frequency state model

This continuous frequency distributed model is not directly usable. A practical model is obtained by frequency discretization of $\mu(\omega)$, where the function $\mu(\omega)$ is replaced by a multiple step function (with J steps).

For an elementary step, its height is $\mu(\omega_k)$ and width $\Delta\omega_k$. Let c_k be the weight of the kth element

$$c_k = \mu(\omega_k)\Delta\omega_k \tag{17}$$

Thus, the continuous distributed model becomes a conventional state model with dimension equal to J.

$$\begin{cases} \dfrac{dz_k(t)}{dt} = -\omega_k z_k(t) + v(t) \big| k = 1..J \\ x(t) = \displaystyle\sum_{k=1}^{J} \mu(\omega_k) z_k(t) \Delta\omega_k \\ \quad\;\; = \displaystyle\sum_{k=1}^{J} c_k z_k(t) \end{cases} \tag{18}$$

or equivalently

$$\frac{d\underline{Z}(t)}{dt} = A_I \underline{Z}(t) + \underline{B}_I v(t)$$
$$x(t) = \underline{C}_I^T \underline{Z}(t) \tag{19}$$

with

$$\underline{Z}(t) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_J \end{bmatrix} \text{ and } A_I = \begin{bmatrix} -\omega_1 & & 0 \\ & \ddots & \\ 0 & & -\omega_J \end{bmatrix} \tag{20}$$

$$\underline{B}_I^T = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}; \quad \underline{C}_I^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_J \end{bmatrix} \tag{21}$$

With this time approach, we get a modal state model of $I^n(s)$ with the requirements $\omega_1 \to 0$, $\omega_J \to \infty$ and $J \ll 1$.

4. State space model of FDEs

4.1. Introduction

The association of the pseudo-state model of the FDE (see Appendix 1) and of the state model of each fractional integrator leads naturally to the global state model of the FDE, which is an equivalent ODE, with infinite dimension [23].

4.2. State space model of an FDE

The state model is based on the pseudo state model of the FDE (63) (65), with input $u(t)$, output $y(t)$ and pseudo state variable $\underline{X}(t)$ whose dimension is N, where N is the number of fractional derivatives (or equivalently the number of fractional integrators).

Then

$$D^{\underline{n}}(\underline{X}(t)) = A\underline{X}(t) + \underline{B}u(t)$$
$$y(t) = \underline{C}^T \underline{X}(t) \tag{22}$$

$$\underline{X}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}; \quad \underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix}; \quad D^{\underline{n}}(\underline{X}(t)) = \begin{bmatrix} D^{n_1}(x_1(t)) \\ D^{n_2}(x_2(t)) \\ \vdots \\ D^{n_N}(x_N(t)) \end{bmatrix}$$

A, $\underline{B}$ and $\underline{C}$ (64) (66) have been expressed in the controller canonical form, but it would be possible to use other canonical forms [11].

The components $x_i(t)$ of the pseudo state vector are the outputs of the N fractional integrators $I^{n_i}(s)$, their inputs $v_i(t)$ depending on the chosen form of the pseudo state model.

In this paper, we assume (only for simplicity) that $0 < n_i < 1 \; \forall i$.

According to the definitions of Section 3, there are two possible models for the fractional integrators.

4.2.1. Continuous frequency distributed state

Let $z_i(\omega,t)$ be the continuously distributed state of $I^{n_i}(s)$, verifying the following state model:

$$\begin{cases} \dfrac{\partial}{\partial t} z_i(\omega,t) = -\omega z_i(\omega,t) + v_i(t) \\ x_i(t) = \int_0^\infty \mu_i(\omega) z_i(\omega,t)\, d\omega \end{cases} \tag{23}$$

with

$$\mu_i(\omega) = \frac{\sin(n_i\pi)}{\pi}\omega^{-n_i} \tag{24}$$

$\mu_i(\omega)$ is the frequency weighting function of the state variable $z_i(\omega,t)$ with the fractional order n_i.

This state space model (23), (24) and closely related concepts have been discussed by a number of authors before: refer to the survey paper by Diethelm [6] and the references cited there, in particular the works of Chatterjee [4] and Yuan and Agrawal [24].

4.2.2. Discrete frequency distributed state

Let $\underline{Z}_i(t)$ be the discrete frequency distributed state vector ($\dim \underline{Z}_i(t) = J$); $\underline{Z}_i(t)$ verifies the following state equation:

$$\dot{\underline{Z}}_i(t) = A_{I_i} \underline{Z}_i(t) + \underline{B}_{I_i} v_i(t)$$
$$x_i(t) = \underline{C}_{I_i}^T \underline{Z}_i(t) \tag{25}$$

where A_{Ii}, $\underline{B}_{I_i}$ and $\underline{C}_{I_i}$ correspond to the fractional order n_i.

In the case of the pseudo state model in controller canonical form, the input $v_i(t)$ of each fractional integrator verifies the following relations:

$$v_i(t) = x_{i+1}(t) \quad (i = 1 \text{ to } N-1)$$

$$v_i(t) = u(t) - \sum_{i=0}^{N-1} a_i x_{i+1}(t) \quad (i = N) \tag{26}$$

4.3. Comments

The pseudo state variables $x_i(t)$ are the outputs of the fractional integrators $I^{n_i}(s)$. In the frequency discrete case, $x_i(t)$ is defined as $x_i(t)=\underline{C}_{I_i}^T\underline{Z}_i(t)$.

This means that $x_i(t)$ is the weighted sum of the components $z_{i,j}(t)$ $(i=0,\ldots,J)$ of the state vector $\underline{Z}_i(t)$ of the considered integrator.

The variables $z_{i,j}(t)$ are true state variables corresponding to the outputs of first order systems (18), they are able to memorize an initial condition.

On the other hand, because $x_i(t)$ is the weighted sum of these state variables, it is not a true state variable because it is not intrinsically able to memorize an initial condition.

The true state vector of the FDE is composed of all the states of the different fractional integrators: thus the state vector of an FDE is infinite dimensional, even when there is only one fractional derivative in the FDE.

The FDE has been converted into an exactly equivalent infinite dimensional ODE (in the case of the continuously distributed state). This is not due to an approximation of the FDE by an integer model, but due to the fundamental property of the fractional integrator which is intrinsically of integer nature with infinite distributed dimension.

4.4. Example: one derivative FDE

Though it is an elementary model, the one derivative FDE is an important case for testing the application of Lyapunov's technique to FDEs.

Consider

$$H(s)=\frac{1}{s^n+a} \tag{27}$$

or

$$D^n(x(t))+ax(t)=u(t) \tag{28}$$

This system is characterized by the continuous frequency distributed model

$$\frac{\partial z(\omega,t)}{\partial t}=-\omega z(\omega,t)+u(t)-ax(t)$$
$$x(t)=\int_0^\infty \mu(\omega)z(\omega,t)d\omega \tag{29}$$

with

$$\mu(\omega)=\frac{\sin n\pi}{\pi}\omega^{-n}$$

or by the discrete frequency distributed model

$$\frac{d\underline{Z}}{dt}=A_I\underline{Z}+\underline{B}_I(u(t)-ax(t))$$
$$x(t)=\underline{C}_I^T\underline{Z}$$
$$\dim\underline{Z}=J \tag{30}$$

This last model can be expressed as

$$\frac{d\underline{Z}}{dt}=A_I\underline{Z}+\underline{B}_I(u(t)-a\underline{C}_I^T\underline{Z})\frac{d\underline{Z}}{dt}=A^*\underline{Z}+\underline{B}_Iu(t)x(t)=\underline{C}_I^T\underline{Z} \tag{31}$$

with

$$A^*=A_I-a\underline{B}_I\underline{C}_I^T \tag{32}$$

A_I is a diagonal matrix while A^* is a full matrix, with $\dim\underline{A}^*=J*J$.

5. Direct Lyapunov approach

5.1. Application of Lyapunov's technique to FDEs

The application of Lyapunov's technique relies on the definition of a Lyapunov function.

With linear ODEs, it is well known that this function $V(\underline{X}(t))$ has to be a quadratic form [11]

$$V(t)=\underline{X}^TP\underline{X} \tag{33}$$

where $\underline{X}$ is the state of the system and P a positive definite matrix and $V(t)$ represents the energy of the system.

According to Lyapunov's theory [14], the system is stable if $(dV(t)/dt)<0$, i.e. if its energy decreases (for an autonomous system with no input).

Consider now the FDE case corresponding to

$$D^{\underline{n}}(\underline{X}(t))=A\underline{X}(t) \tag{34}$$

where $\underline{X}(t)$ is a pseudo state vector.

The first step is to define a Lyapunov function $V(\underline{X}(t))$. Is it realistic to use a quadratic function of $\underline{X}(t)$ in the fractional case ?

Moreover, how is it possible to characterize the system energy decrease, assuming that we have been able to define $V(\underline{X}(t))$?

The choice $dV(t)/dt$is adapted to the ODE case. In the FDE case, perhaps it would be more convenient to use a fractional derivative of $V(t)$?

For example, with a commensurate order system, $D^n(V(\underline{X}))$ appears to be a possible choice; nevertheless, is $D^n(V(\underline{X}))$ adapted to characterize energy decrease?

Let us notice that in the noncommensurate case, there are N values n_i, i.e. N possible fractional derivatives!

This brief discussion illustrates the difficulty to apply Lyapunov's technique to FDEs.

5.2. Direct approach

Fortunately, the fractional integrator concept ($I^{n_i}(s)$, with internal state $\underline{Z}_i$ or $z_i(\omega,t)$) provides a realistic solution to these difficulties.

In fact, we have shown that $\underline{X}$ is only a pseudo state vector that has to be replaced by the state $\underline{Z}$ (in the discrete frequency distributed case).

Thus, the FDE is equivalent to an ODE with integer order derivatives, characterized by an infinite dimensional state (large dimensional in practice).

Effectively, if the system is composed of N fractional integrators, $\dim\underline{Z}=NJ$ with $J\ll 1$.

In spite of this difficulty linked to the dimension of $\underline{Z}$, the FDE has been transformed into an equivalent ODE, which can be characterized by its Lyapunov function $V(\underline{Z})$ characterizing the system energy.

Consider the one derivative FDE case (31) (32):

$$V(t)=V(\underline{Z})=\underline{Z}^TP\underline{Z} \tag{35}$$

where P is a positive definite matrix.

Because $(d\underline{Z}/dt)=A^*\underline{Z}$ then

$$\frac{dV(t)}{dt}=\underline{Z}^T(A^{*T}P+PA^*)\underline{Z}. \tag{36}$$

The equivalent ODE is stable if $(dV(t)/dt)<0$, i.e. if $A^{*T}P+PA^*$ is a negative definite matrix.

The equation

$$A^{*T}P+PA^*=-I \tag{37}$$

defines a linear system whose solutions are the p_{ij} coefficients of matrix P (with $p_{ij}=p_{ji}$).

Practically, for a given A^*, Eq. (37) provides the matrix P and the FDE is stable if P is positive definite, i.e. if all its minors are positive.

Apparently, there is no difficulty to apply Lyapunov's technique to FDEs.

But this apparent simplicity is an illusion.

In fact, $\dim P=\dim A^*=J*J$ with $J\ll 1$.

Thus, it is necessary to calculate the coefficients p_{ij} with a large number of equations and then to test the positivity of all the minors of P, it is obvious that numerical problems will arise with this direct approach.

Indeed, these numerical problems will be much more difficult to solve with an N derivatives FDE, thus the direct approach does not seem to be realistic.

6. Indirect Lyapunov approach

6.1. Linear FDE

This indirect approach is based on the continuous frequency distributed model of the fractional integrator.

Consider the one derivative case

$$D^n(x)+ax=0 \tag{38}$$

which is exactly equivalent to the ODE

$$\frac{\partial z(\omega,t)}{\partial t}=-\omega z(\omega,t)-ax(t)$$

$$x(t)=\int_0^\infty \mu(\omega)z(\omega,t)\,d\omega \tag{39}$$

with

$$\mu(\omega)=\frac{\sin n\pi}{\pi}\omega^{-n}$$

Let us define two Lyapunov functions

- $v(\omega,t)$ is the monochromatic Lyapunov function corresponding to the elementary frequency ω.
- $V(t)$ is the Lyapunov function summing all the monochromatic $v(\omega,t)$ with the weighting function $\mu(\omega)$.

Thus

$$v(\omega,t)=z^2(\omega,t) \tag{40}$$

and

$$V(t)=\int_0^\infty \mu(\omega)v(\omega,t)\,d\omega=\int_0^\infty \mu(\omega)z^2(\omega,t)\,d\omega \tag{41}$$

Indeed, $v(\omega,t)$ is positive. Because $\mu(\omega)$ is positive for all ω, $V(t)$ is also a positive Lyapunov function.

Then

$$\frac{\partial v(\omega,t)}{\partial t}=-2\omega z^2(\omega,t)-2az(\omega,t)x(t) \tag{42}$$

and

$$\frac{dV(t)}{dt}=\int_0^\infty \mu(\omega)\frac{\partial v(\omega,t)}{\partial t}\,d\omega$$

$$\frac{dV(t)}{dt}=-2\int_0^\infty \mu(\omega)\omega z^2(\omega,t)\,d\omega-2ax(t)\int_0^\infty \mu(\omega)z(\omega,t)\,d\omega \tag{43}$$

Finally

$$\frac{dV(t)}{dt}=-2\int_0^\infty \mu(\omega)\omega z^2(\omega,t)\,d\omega-2ax^2(t) \tag{44}$$

Owing to the Lemma of Appendix 2, $dV(t)/dt$ is negative if $a>0$.

Conclusion: the one derivative FDE

$$D^n(x)+ax=0 \text{ is stable if } a>0 \tag{45}$$

which is a well known result in the linear case [17].

Remark 1. With the direct approach, it is necessary to solve a large dimensional problem (37) in order to investigate the stability of an elementary FDE. So, it is obvious that the indirect approach leads to a parsimonious methodology, which is the necessary requirement to solve more complex problems.

Remark 2. A complete analysis of the considered FDE stability would need to investigate the complex parameter case. Unfortunately, this case needs more sophisticated tools and it is not possible to conclude with the presented theory (see Appendix 3).

6.2. Nonlinear FDE

Consider the nonlinear FDE

$$D^n(x)=f(x) \tag{46}$$

with $f(x)=ax^3+bx$ where $a>0$ and $b<0$.

Owing to the continuous frequency distributed model of the fractional integrator, the nonlinear system can be expressed as

$$\frac{\partial z(\omega,t)}{\partial t}=-\omega z(\omega,t)+f(x(t))$$

$$x(t)=\int_0^\infty \mu(\omega)z(\omega,t)\,d\omega \tag{47}$$

with

$$\mu(\omega)=\frac{\sin n\pi}{\pi}\omega^{-n}$$

Because this system is nonlinear, the definition of the Lyapunov function will be performed using the variable gradient method [7,12].

Successively, we have to calculate $\partial v(\omega,t)/\partial t$, $dV(t)/dt$ and finally $v(\omega,t)$ and $V(t)$.

Consider the monochromatic Lyapunov function $v(\omega,t)$.

Let us define

$$\frac{\partial v(\omega,t)}{\partial z} = \alpha z \tag{48}$$

then

$$\frac{\partial v(\omega,t)}{\partial t} = \frac{\partial v}{\partial z}\frac{\partial z}{\partial t} = \alpha z(-\omega z + f(x)) \tag{49}$$

and

$$\frac{\partial v(\omega,t)}{\partial t} = -\omega\alpha z^2(\omega,t) + \alpha z(\omega,t)(ax^3(t)+bx(t)) \tag{50}$$

Then

$$\frac{dV(t)}{dt} = \int_0^\infty \mu(\omega)\frac{\partial v(\omega,t)}{\partial t}\,d\omega \tag{51}$$

and finally

$$\frac{dV}{dt} = -\alpha\int_0^\infty \mu(\omega)\omega z^2(\omega,t)\,d\omega + \alpha x^2(t)(ax^2(t)+b) \tag{52}$$

Then, let us consider $v(\omega,t)$and $V(t)$. Because we have defined $(\partial v(\omega,t))/\partial z = \alpha z$, we get $v(\omega,t)$ by integration along the path joining the origin to z

$$v(\omega,t) = \int_0^z \frac{\partial v}{\partial u}\,du = \int_0^z \alpha u\,du = \alpha\frac{z^2}{2} \tag{53}$$

Finally,

$$V(t) = \int_0^\infty \mu(\omega)v(\omega,t)\,d\omega = \alpha\int_0^\infty \mu(\omega)\frac{z^2(\omega,t)}{2}\,d\omega \tag{54}$$

Stability condition: The considered system is stable if $V(t)>0$ and $(dV(t)/dt)<0$. $V(t)$ is positive if $\alpha>0$. Owing to the lemma of Appendix 2, $dV(t)/dt$ is negative if

$$\alpha x^2(ax^2+b)<0, \quad \text{i.e. if } \; ax^2+b<0 \tag{55}$$

This last condition defines the stability domain:

$$-\sqrt{\frac{-b}{a}} < x < \sqrt{\frac{-b}{a}} \tag{56}$$

7. Conclusion

A Lyapunov approach to the stability of fractional differential equations has been presented in this paper. The key concept of this methodology is the frequency distributed fractional integrator model, which is the basis of a global state space model of FDEs.

Because this global model is an equivalent ODE, usual Lyapunov tools can be used to analyze the stability of both linear and nonlinear FDEs. The infinite dimension problem arising with the direct approach has been solved using the continuous frequency distributed model, with specific Lyapunov functions, which leads to parsimonious parametric conditions, fundamental requirement for the analysis of N derivatives FDEs.

Only the concepts of this new methodology have been presented in this paper. Further research will be focused on N derivatives FDEs and their LMI stability conditions.

Appendix 1. Simulation of an FDE and its pseudo state model

A.1. Simulation of an FDE

Consider the FDE (9).

Define

$$X(s) = \frac{1}{A(s)}U(s) \tag{57}$$

and

$$Y(s) = B(s)X(s) \tag{58}$$

which permits to introduce the classical controller canonical state space form [11]

$$\begin{aligned} x_1(t) &= x(t)\\ x_2(t) &= D^{n_1}(x_1(t))\\ &\vdots\\ x_i(t) &= D^{n_{i-1}}(x_{i-1}(t))\\ &\vdots\\ x_N(t) &= D^{n_{N-1}}(x_{N-1}(t))\\ D^{n_N}(x_N(t)) &= -a_0x_1(t)\cdots - a_{N-1}x_N(t) + u(t) = \varepsilon(t) \end{aligned} \tag{59}$$

and

$$\begin{aligned} x_1(t) &= I^{n_1}(x_2(t))\\ &\vdots\\ x_{i-1}(t) &= I^{n_{i-1}}(x_i(t))\\ &\vdots\\ x_{N-1}(t) &= I^{n_{N-1}}(x_N(t))\\ x_N(t) &= I^{n_N}(\varepsilon(t)) \end{aligned} \tag{60}$$

This simulation scheme is based on a state space model which requires N fractional integration operators, whose transfer functions are, respectively, $\{I^{n_N}(s), I^{n_{N-1}}(s), \ldots, I^{n_1}(s)\}$ and connected according to the analog simulation scheme of Fig. 1.

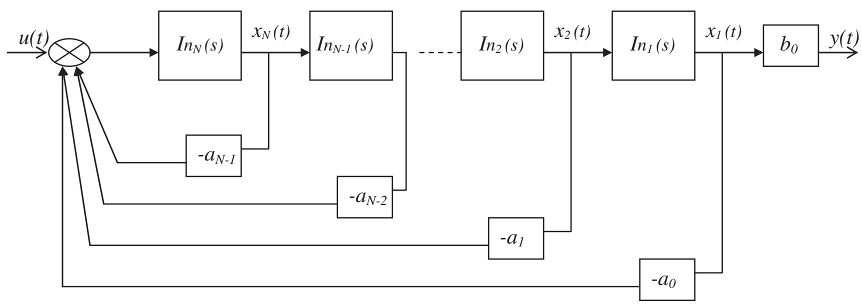

Fig. 1. Simulation of an FDE with fractional integrators.

Finally, $y(t)$ is obtained using the relation

$$Y(s) = B(s)X(s) \tag{61}$$

Corresponding to

$$y(t) = \sum_{i=0}^{M-1} b_i x_{i+1}(t) \tag{62}$$

A.2. Pseudo state-space model of the FDE

FDE simulation is based on a fractional state-space model which can be expressed as

$$D^{\underline{n}}(\underline{X}(t)) = A\underline{X}(t) + \underline{B}u(t) \tag{63}$$

with

$$\underline{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_i(t) \\ \vdots \\ x_N(t) \end{bmatrix} \text{ and } D^{\underline{n}}(\underline{X}(t)) = \begin{bmatrix} D^{n_1}(x_1(t)) \\ \vdots \\ D^{n_i}(x_i(t)) \\ \vdots \\ D^{n_N}(x_N(t)) \end{bmatrix} \tag{64}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_N \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The output is

$$y(t) = \underline{C}^T \underline{X}(t) \tag{65}$$

with

$$\underline{C}^T = \begin{bmatrix} b_0 & \dots & b_M & 0 & \dots & 0 \end{bmatrix} \tag{66}$$

These relations define the pseudo state space model of the FDE in controller canonical form [11].

Appendix 2. Lemma

Consider

$$W = W_1 + aW_2 \tag{67}$$

with

$$W_1 = \int_0^\infty \mu(\omega)\omega z^2(\omega,t)\, d\omega \tag{68}$$

and

$$W_2 = x^2(t) \tag{69}$$

The frequency discretization of W_1 gives

$$W_1 = \sum_{j=1}^{J} \omega_j \mu(\omega_j) z^2(\omega_j,t) \Delta\omega_j = \sum_{j=1}^{J} \omega_j c_j Z_j^2 \tag{70}$$

Because $\omega_j c_j > 0$ for all j, W_1 is a positive definite quadratic form. Moreover, W_1 can be expressed in the matrix form as

$$W_1 = \underline{Z}^T M \underline{Z} \tag{71}$$

with

$$M = \begin{bmatrix} \omega_1 c_1 & 0 & 0 \\ 0 & \omega_j c_j & 0 \\ 0 & 0 & \omega_J c_J \end{bmatrix} \tag{72}$$

According to its definition (69), W_2 is positive. Because $x = \underline{C}^T \underline{Z}$, W_2 can be expressed as

$$W_2 = \underline{Z}^T \underline{C}\,\underline{C}^T \underline{Z} \tag{73}$$

This second expression of W_2 is only positive semi-definite. Consequently, $W = W_1 + aW_2$ is a positive definite quadratic form if

$$M + a\underline{C}\,\underline{C}^T \tag{74}$$

is a positive definite matrix, i.e. if all its minors are positive. Indeed, because W_1 and W_2 are positive, a positive value of a provides $W > 0$. But, is it possible to find a negative value of a satisfying the positivity of W?

Theoretically, it would be necessary to test all the minors of (74). Practically, a response is given by the first minor

$$m_1 = \omega_1 c_1 + a c_1^2 \tag{75}$$

$$m_1 \text{ is positive if } a > -\frac{\omega_1}{c_1} \tag{76}$$

But because a necessary requirement (Section 4.2) is $\omega_1 \to 0$, m_1 is positive if $a > 0$.

Lemma. *The quadratic form $W = W_1 + aW_2$ is positive definite if and only if $a > 0$.*

Appendix 3. Stability anamysis xith complex parameter

Consider the FDE

$$D^n(f(t)) + af(t) = 0 \tag{77}$$

where $a = \alpha + j\beta$ is a complex parameter.

Then, $f(t)$ is a complex function

$$f(t) = x(t) + jy(t) \tag{78}$$

The original FDE is equivalent to two coupled FDEs

$$\begin{aligned} D^n(x(t)) + \alpha x(t) - \beta y(t) = 0 \\ D^n(y(t)) + \alpha y(t) + \beta x(t) = 0 \end{aligned} \tag{79}$$

C.1. Conventional stability analysis

Because of coupling, the stability of the real (x) and imaginary (y) components is subject to the same characteristic polynomial

$$s^{2n} + 2\alpha s^n + \alpha^2 + \beta^2 = 0 \tag{80}$$

We have analyzed the stability of this polynomial using a frequency approach [22] for n=0.5

- for $\alpha > 0$ the system is unconditionally stable
- for $\alpha < 0$ the system is stable if

$$\beta^2 > \alpha^2 \tag{81}$$

Using Matignon's result [15], the transfer function $1/(s^n - \lambda)$ is stable if its pole lays outside the domain defined by the condition

$$|\arg(\lambda)| \leq n\frac{\pi}{2} \tag{82}$$

With the definition $\lambda = \lambda_x + j\lambda_y$ and $n = 0.5$, we get the same stability domain as previously, with $\lambda_x = -\alpha$ and $\lambda_y = -\beta$

C.2. Application of Lyapunov's approach

Because a is complex, the system is governed not only by one variable $f(t)$ but also by two pseudo state space variables $x(t)$ and $y(t)$, with their own internal distributed variables, $z_x(\omega,t)$ and $z_y(\omega,t)$, respectively.

So, we have to define two monochromatic Lyapunov functions

$$v_x(\omega,t) = z_x^2(\omega,t) \text{ and } v_y(\omega,t) = z_y^2(\omega,t) \tag{83}$$

Because the variables $x(t)$ and $y(t)$ are coupled, their internal variables $z_x(\omega,t)$ and $z_y(\omega,t)$ are also coupled.

Thus, $v(\omega,t)$ is necessarily a quadratic form weighted by a P matrix, which has to be symmetric and positive definite

$$v(\omega,t) = \underline{z}^T P \underline{z} \tag{84}$$

with

$$\underline{z} = \begin{bmatrix} z_x \\ z_y \end{bmatrix} \text{ and } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{85}$$

Thus

$$v(\omega,t) = p_{11}z_x^2 + 2p_{12}z_xz_y + p_{22}z_y^2 \tag{86}$$

Finally, the global Lyapunov function $V(t)$ is obtained by the summation of all the monochromatic functions $v(\omega,t)$ on the $[0,\infty[$ range with the weighting function $\mu(\omega)$.

So, the problem is more complex than with only one monochromatic function $v(\omega,t)$. Moreover, the solution is given by a set of LMI conditions which are not appropriate to formulate an analytical expression of the stability domain.

Remark: it would be interesting to express $v(\omega,t)$ as

$$v(\omega,t) = z(\omega,t)\bar{z}(\omega,t) \tag{87}$$

with

$$z(\omega,t) = z_x + jz_y$$

then, we would get

$$v(\omega,t) = z_x^2 + z_y^2 \tag{88}$$

and

$$V(t) = \int_0^\infty \mu(\omega)z_x^2(\omega,t)\,d\omega + \int_0^\infty \mu(\omega)z_y^2(\omega,t)\,d\omega = \int_0^\infty \mu(\omega)|z(\omega,t)|^2\,d\omega \tag{89}$$

which is a more simple expression than previously.

This choice of $V(t)$ leads to the derivative

$$\frac{dV(t)}{dt} = -2\int_0^\infty \mu(\omega)\omega|z(\omega,t)|^2\,d\omega - 2\alpha|f(t)|^2 \tag{90}$$

which gives the condition

$$\frac{dV(t)}{dt} < 0 \text{ if } \alpha > 0 \tag{91}$$

This result is correct, but conservative, because it does not take into account the domain defined by $\beta^2 > \alpha^2$ for $\alpha < 0$.

The conservatism is caused by the choice $v(\omega,t) = z_x^2 + z_y^2$ which is wrong.

An appropriate Lyapunov function has to take into account the coupling of the internal variables with the matrix P.

C.3. Conclusion

This stability analysis requires a Lyapunov function with two coupled variables, which is outside the initial objective of the paper.

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